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The Surface Charge of a Semi-Infinite Cylinder
Due to an Axial Point Charge

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THE SURFACE CHARGE OF A SEMI-INFINITE
CYLINDER DUE TO AN AXIAL POINT CHARGE*)

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Summary

In this paper we determine the surface charge density of a semi-infinite conducting cylinder due to an axial point charge. The problem can be reduced to an integral equation of the Wiener-Hopf type. The relevant factorization problem is studied in detail.

§ 1. *Introduction.* The problem of the determination of the field of an axial point charge inside a hollow infinitely long conducting cylinder has been considered by various writers¹⁾.

If there is a point charge -1 at the origin and if in cartesian coordinates (x, y, z) the position of the cylinder is determined by $-\infty < x < \infty$, $r = \sqrt{y^2 + z^2} = 1$ the surface charge at the cylinder is

$$\sigma(x) = \frac{1}{2\pi^2} \int_0^{\infty} \frac{\cos tx}{I_0(t)} dt. \quad (1.1)$$

In this paper we shall consider the equivalent problem for a semi-infinite cylinder $0 < x < \infty$, $r = 1$. This problem is much more complicated. The determination of the surface charge $\sigma(x)$, which also depends on the position of the point charge on the axis, involves the solving of a Wiener-Hopf equation of the first kind

$$\int_0^{\infty} h(x-t) \sigma(t) dt = g(x), \quad x > 0 \quad (1.2)$$

with $h(x) \sim -\ln x^2$ for $x \rightarrow 0$. If $\sigma(x)$ is known, the electrostatic

*) This paper is a revised version of a solution of a prize question put by the Wiskundig Genootschap in 1955.

field $V(r, x)$ may be easily determined (cf. (2.3)). We shall therefore confine our attention to the determination of $\sigma(x)$ from (1.2).

The Wiener-Hopf equation (1.2) cannot be solved by the familiar methods since in this case the strip of convergence is absent. There is only a line of convergence, the real axis, upon which the Fourier transform of the kernel function has a logarithmic singularity at the origin. However, the integral equation (1.2) may be solved by means of the methods developed by Muskhelishvili ²⁾; cf. also ³⁾ and ⁴⁾. In particular the notion of sectionally holomorphic functions plays an important role in the solution of (1.2) ⁵⁾. In § 3 this will be discussed in detail. An explicit expression is obtained for the Fourier transform of the surface charge $\sigma(x)$ depending on the proper factorization of the Fourier transform $H(x)$ of $h(x)$.

The explicit factorization is carried out in § 4. Various integral representations and series expansions are derived. This may lead to an expansion of $\sigma(x)$ for small x in § 5, or for large x in § 7. If the point charge is at the open end $(0, 0)$ of the cylinder $0 < x < \infty$, $r = 1$, we have e.g.

$$2\pi\sigma(x) = 0.337x^{-1/2} + 0.527x^{1/2} - 0.782^{3/2} - 0.961^{5/2} \dots \quad (1.3)$$

Special expressions are derived in § 6 for the case that the point charge is far inside or outside the cylinder.

§ 2. *Reduction of the problem to an integral equation.* The electrostatic field $V(r, x)$ of a point charge at $(0, a)$ of intensity -1 in the presence of a semi-infinite conducting cylinder $r = 1$, $0 < x < \infty$ is determined by

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial x^2} = 0, \quad (2.1)$$

$$V = 0 \quad \text{for } r = 1, x > 0,$$

$$V = -[r^2 + (x - a)^2]^{-1/2} + O(1) \quad \text{for } r^2 + (x - a)^2 \rightarrow 0.$$

The induced charge density at the cylinder is

$$\sigma(x) = -\frac{1}{4\pi} \frac{\partial V}{\partial r} \bigg|_{r=1}^{1+0}. \quad (2.2)$$

Conversely, if $\sigma(x)$ is known, the potential $V(r, x)$ may be deter-

mined from

$$V(r, x) = -[r^2 + (x - a)^2]^{-\frac{1}{2}} + \int_0^\infty \sigma(t) dt \int_0^{2\pi} [r^2 + 1 + 2r \cos \theta + (x - t)^2]^{-\frac{1}{2}} d\theta$$

or

$$V(r, x) = -[r^2 + (x - a)^2]^{-\frac{1}{2}} + \int_0^\infty h(r, x - t) \sigma(t) dt, \quad (2.3)$$

where

$$h(r, x) = 4[(r + 1)^2 + x^2]^{-\frac{1}{2}} K\{2r^{\frac{1}{2}}[(r + 1)^2 + x^2]^{-\frac{1}{2}}\}. \quad (2.4)$$

The function $K(k)$ is the complete elliptic integral

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi = \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}; 1; k^2).$$

For $r = 1$, $x > 0$ from (2.3) the following integral equation for $\sigma(x)$ is obtained:

$$\int_0^\infty h(x - t) \sigma(t) dt = g(x), \quad (2.5)$$

where

$$h(x) = 4(x^2 + 4)^{-\frac{1}{2}} K[2(x^2 + 4)^{-\frac{1}{2}}], \quad (2.6)$$

and

$$g(x) = [1 + (x - a)^2]^{-\frac{1}{2}}. \quad (2.7)$$

Equation (2.5) is of the well-known Wiener-Hopf type. In view of

$$h(x) \sim -\ln x^2, \quad x \rightarrow 0$$

the integral equation is singular.

§ 3. *Solution of the integral equation.* By $F^+(z)$ we shall understand a function of the complex variable $z = x + iy$ which is holomorphic in the upper half-plane $y > 0$ and which vanishes at infinity. In a similar way $F^-(z)$ is holomorphic in the lower half-plane $y < 0$ and vanishes at infinity. At the real axis we define

$$F^+(x) = \lim_{y \downarrow 0} F^+(x + iy) \quad F^-(x) = \lim_{y \uparrow 0} F^-(x + iy).$$

Consider now the Wiener-Hopf equation (2.5) in the form

$$\int_0^\infty h(x - t) \sigma(t) dt = \begin{cases} g(x) & x > 0, \\ \chi(x) & x < 0, \end{cases} \quad (3.1)$$

where $\sigma(x)$, $x > 0$ and $\chi(x)$, $x < 0$ are unknown functions. Let $H(x)$, $S^+(z)$, $G^+(z)$, $X^-(z)$ represent the following Fourier transforms:

$$\begin{aligned} H(x) &= \int_{-\infty}^{\infty} e^{itx} h(t) dt, \\ S^+(z) &= \int_0^{\infty} e^{itz} \sigma(t) dt, \quad G^+(z) = \int_0^{\infty} e^{itz} g(t) dt, \\ X^-(z) &= \int_{-\infty}^0 e^{itz} \chi(t) dt, \end{aligned}$$

then Fourier transformation of (3.1) gives

$$H(x)S^+(x) = G^+(x) + X^-(x). \quad (3.2)$$

The kernel function $H(x)$ can be factorized as the product of limit functions of sectionally holomorphic functions $H^+(z)$ and $H^-(z)$:

$$H(x) = H^+(x) H^-(x). \quad (3.3)$$

Then (3.2) may be written as follows:

$$H^+(x) S^+(x) - \frac{X^-(x)}{H^-(x)} = \frac{G^+(x)}{H^-(x)}. \quad (3.4)$$

If the factorization (3.3) is carried out properly, the relation (3.4) is of the form

$$\phi^+(x) - \phi^-(x) = \varphi(x). \quad (3.5)$$

The solution of the problem (3.5) is unique if ϕ^+ , ϕ^- and φ belong to any finite L^2 -class, i.e. for $-A < x < A$ for any positive A . We have

$$\phi^{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - z} dt, \quad (3.6)$$

and

$$\phi^+(x) = \frac{1}{2} \varphi(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x} dt, \quad (3.7)$$

where

$$\int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x} dt \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x+t) - \varphi(x-t)}{t} dt. \quad (3.8)$$

Thus the following explicit solution of (3.2) is obtained:

$$S^+(z) = \frac{1}{2\pi i H^+(z)} \int_{-\infty}^{\infty} \frac{G^+(t)}{H^-(t)} \frac{dt}{t-z}. \quad (3.9)$$

From $S(z)$ the surface charge $\sigma(x)$ may be derived by inverse Fourier transformation.

In view of

$$\int_{-\infty}^{\infty} \phi^-(t) \frac{dt}{t-z} = 0 \quad \text{for } \operatorname{Im} z > 0$$

the right-hand side of (3.9) may be replaced by

$$S^+(z) = \frac{1}{2\pi i H^+(z)} \int_{-\infty}^{\infty} \frac{G(t)}{H^-(t)} \frac{dt}{t-z}, \quad (3.10)$$

where

$$G(x) = \int_{-\infty}^{\infty} e^{itx} g(t) dt.$$

We have

$$G(x) = 2e^{iux} K_0(|x|) \quad (3.11)$$

and

$$H(x) = 4\pi I_0(x) K_0(|x|), \quad (3.12)$$

cf. Erdélyi, Tables of Integral Transforms (1.14.13).

§ 4. *Factorization of $H(x)$.* Introduce the function

$$L(x) \stackrel{\text{def}}{=} \frac{|x|}{2\pi} H(x) = 2|x| I_0(x) K_0(|x|). \quad (4.1)$$

For $x \rightarrow \infty$ we have

$$\ln L(x) = \frac{1}{8x^2} + \frac{13}{64x^4} + O(x^{-6}). \quad (4.2)$$

Then the factorization of $H(x)$ is obtained as follows

$$H^+(z) = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \exp \left[\frac{\pi i}{4} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t-z} dt \right], \quad \operatorname{Im} z \geq 0, \quad (4.3)$$

$$H^-(z) = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \exp \left[-\frac{\pi i}{4} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t-z} dt \right], \quad \text{Im } z \leq 0. \quad (4.4)$$

From (4.3) we obtain for $z \rightarrow x$

$$H^+(x) = 2 [\pi I_0(x) K_0(|x|)]^{\frac{1}{2}} \exp \left[\frac{\pi i}{4} \text{sgn } x + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t-x} dt \right]; \quad (4.5)$$

also

$$H^-(x) = \overline{H^+(x)}, \quad (4.6)$$

$$\left. \begin{aligned} H^+(xe^{\pi i}) &= H^-(x), \\ H^-(xe^{-\pi i}) &= H^+(x), \end{aligned} \right\} \quad \text{for } x > 0. \quad (4.7)$$

The asymptotic behaviour of $H^+(z)$ may be determined as follows. We have

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t-z} dt = \sum_1 \frac{\alpha_k}{s^k}$$

where $s = -iz$, $\text{Re } s > 0$. The first few coefficients are

$$\begin{aligned} \alpha_1 &= \frac{1}{\pi} \int_0^{\infty} \ln L(t) dt = -0.0280, \quad \alpha_2 = -\frac{1}{16}, \\ \alpha_3 &= -\frac{1}{\pi} \int_0^{\infty} t^2 \left[\ln L(t) - \frac{1}{8t^2} \right] dt = -0.00298, \quad \alpha_4 = \frac{13}{128}. \end{aligned}$$

Hence we have

$$z \rightarrow \infty \quad H^+(z) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \exp \sum_1 \frac{\alpha_k}{s^k}, \quad s = -iz. \quad (4.8)$$

For $H^+(x)$ an alternative expression may be derived in the following way. We need the auxiliary function

$$\Omega(x) \stackrel{\text{def}}{=} \arg [-Y_0(x) + iJ_0(x)], \quad x > 0, \quad (4.9)$$

with $0 \leq \Omega(x) \leq \pi$. Without proof we mention the following simple properties of this function:

$$\Omega'(x) = \frac{2}{\pi x [J_0^2(x) + Y_0^2(x)]}. \quad (4.10)$$

For $x \rightarrow \infty$ we have

$$\Omega'(x) = 1 + \frac{1}{8x^2} - \frac{25}{128x^4} + O(x^{-6}). \quad (4.11)$$

For $x \rightarrow +0$ we have

$$\Omega(x) = \frac{\pi}{2 \ln(2/x)} + O\left(\ln^{-2} \frac{2}{x}\right). \quad (4.12)$$

At the k th zero β_k of $J_0(x)$, $k \geq 1$, we have

$$\Omega(\beta_k - 0) = 0, \quad \Omega(\beta_k + 0) = \pi. \quad (4.13)$$

We shall now prove the following formula:

$$\arg H^+(x) = \frac{x}{\pi} \int_0^\infty \frac{\Omega(t)}{t^2 + x^2} dt. \quad (4.14)$$

According to (4.5) we have

$$\arg H^+(x) = \frac{\pi}{4} \operatorname{sgn} x - \frac{x}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2 - x^2} dt. \quad (4.15)$$

This odd function is the sine transform of the even function

$$\frac{1}{2y} + \frac{1}{\pi} \int_0^\infty \cos yt \ln L(t) dt, \quad y > 0.$$

This function may be reduced as follows

$$\begin{aligned} & \frac{1}{2y} - \frac{1}{\pi y} \int_0^\infty \sin yt \frac{L'(t)}{L(t)} dt = \\ &= \frac{1}{2y} - \frac{1}{\pi y} \operatorname{Im} \int_0^\infty e^{iyt} \left[\frac{1}{t} + \frac{I_0'(t)}{I_0(t)} + \frac{K_0'(t)}{K_0(t)} \right] dt = \\ &= -\frac{1}{y} \sum_1^\infty e^{-\beta_k y} + \frac{1}{\pi y} \operatorname{Im} \int_0^\infty e^{-yt} \frac{J_0'(t) - iY_0'(t)}{J_0(t) - iY_0(t)} dt = \\ &= -\frac{1}{y} \left[\sum_1^\infty e^{-\beta_k y} - \frac{1}{\pi} \int_0^\infty e^{-yt} \Omega'(t) dt \right] = \frac{1}{\pi} \int_0^\infty e^{-yt} \Omega(t) dt. \end{aligned}$$

The expression last obtained is the sine transform of the righthand side of (4.14). Thus we have

$$H^+(x) = 2[\pi I_0(x) K_0(|x|)]^{\frac{1}{2}} \exp \frac{\pi i}{\pi} \int_0^\infty \frac{\Omega(t)}{t^2 + x^2} dt. \quad (4.16)$$

We note that for a purely imaginary z the function $H^+(z)$ is real. We have from (4.3) and (4.4)

$$H^+(si) = H^-(-si) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \exp \frac{s}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2 + s^2} dt. \quad (4.17)$$

Finally we shall consider the behaviour of $H^+(z)$ near the origin. For $x > 0$ we have

$$\ln L(x) = \ln 2x + \ln \ln \frac{2}{x} + O\left(\ln^{-1} \frac{2}{x}\right).$$

Since

$$\frac{z}{\pi i} \int_0^\infty \frac{\ln 2t}{t^2 - z^2} dt = \frac{1}{2} \ln 2z - \frac{\pi i}{4}, \quad \text{Im } z > 0,$$

and

$$\frac{z}{\pi i} \int_0^\infty \frac{\ln \ln 2t}{t^2 - z^2} dt = \frac{1}{2} \ln \ln \frac{2}{z} + O\left(\ln^{-1} \frac{2}{z}\right), \quad \text{Im } z > 0,$$

we obtain

$$z \rightarrow 0, \quad H^+(z) = 2\left(\pi \ln \frac{2}{z}\right)^{\frac{1}{2}} \left[1 + O\left(\ln^{-1} \frac{2}{z}\right)\right]. \quad (4.18)$$

At the positive real axis we have

$$\begin{aligned} H^+(x) &= 2\left(\pi \ln \frac{2}{x}\right)^{\frac{1}{2}} \left[1 + \frac{\gamma - \frac{1}{2}\pi i}{2 \ln \frac{1}{2}x} + O\left(\ln^{-2} \frac{2}{x}\right)\right], \\ H^-(x) &= 2\left(\pi \ln \frac{2}{x}\right)^{\frac{1}{2}} \left[1 + \frac{\gamma + \frac{1}{2}\pi i}{2 \ln \frac{1}{2}x} + O\left(\ln^{-2} \frac{2}{x}\right)\right]. \end{aligned} \quad (4.19)$$

The argument of $H^+(x)$ as a function of x is given in table I.

TABLE I

x	$\arg. H^+(x)$	x	$\arg. H^+(x)$
0.00	0	1.4	0.739
0.05	0.278	1.6	0.749
0.1	0.349	1.8	0.757
0.15	0.401	2	0.762
0.2	0.441	3	0.772
0.3	0.506	4	0.778
0.4	0.554	5	0.779
0.5	0.593	6	0.780
0.6	0.624	7	0.781
0.8	0.670	8	0.781
1.0	0.701	9	0.782
1.2	0.723	10	0.782
		∞	$\pi/4 = 0.785$

§ 5. *Expansion of $\sigma(x)$ for small x .* The properties of the functions $H^\pm(z)$ derived in the previous section validate the derivation of the solution of the Wiener-Hopf equation (2.5). We have obtained according to (3.10) and (3.11).

$$S^+(z) = \frac{1}{\pi i H^+(z)} \int_{-\infty}^{\infty} e^{iat} \frac{K_0(|t|)}{H^-(t)} \frac{dt}{t-z}, \quad (5.1)$$

or

$$S^+(z) = \frac{1}{4\pi^2 i H^+(z)} \int_{-\infty}^{\infty} e^{iat} \frac{H^+(t)}{I_0(t)} \frac{dt}{t-z}. \quad (5.2)$$

The asymptotic expansion of $S^+(z)$ for $z \rightarrow \infty$ may be determined as follows.

According to (4.8) we have

$$\frac{1}{H^+(z)} = \left(\frac{s}{2\pi} \right)^{\frac{1}{2}} \exp - \sum_1 \frac{\alpha_k}{s^k},$$

where $s = -iz$, $\operatorname{Re} s > 0$. This may be brought into the form

$$\frac{1}{H^+(z)} = \left(\frac{s}{2\pi} \right)^{\frac{1}{2}} \sum_0 \frac{a_k}{s^k}, \quad (5.3)$$

where

$$a_0 = 1, \quad a_1 = -\alpha_1 = 0.0280, \\ a_2 = -\alpha_2 + \frac{\alpha_1^2}{2} = -0.0629, \quad a_3 = -\alpha_3 + \alpha_1\alpha_2 - \frac{\alpha_1^3}{6} = 0.00473.$$

Furthermore we have the asymptotic expansion

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} e^{iat} \frac{K_0(|t|)}{H^-(t)} \frac{dt}{t-z} = 2^{-\frac{1}{2}} \sum_0 \frac{b_k}{s^{k+1}}, \quad (5.4)$$

where

$$b_k = \frac{2^{-\frac{1}{2}}}{\pi} \operatorname{Re} \int_0^{\infty} \exp \left[i \left(\frac{k\pi}{2} - at \right) \right] t^k \frac{K_0(t)}{H^+(t)} dt$$

or, in view of (4.16),

$$b_k = \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_0^{\infty} t^k \left[\frac{K_0(t)}{I_0(t)} \right]^{\frac{1}{2}} \cos \left[\frac{k\pi}{2} - A(t) - at \right] dt, \quad (5.5)$$

where

$$A(x) \stackrel{\text{def}}{=} \arg H^+(x) = \frac{x}{\pi} \int_0^{\infty} \frac{\Omega(t)}{t^2 + x^2} dt. \quad (5.6)$$

By means of (5.3) and (5.4) we obtain from (5.1)

$$2\pi S^+(z) = \sqrt{\pi} \sum_0^{\infty} c_k s^{-k-\frac{1}{2}}, \quad z = si \quad (5.7)$$

where

$$c_k = \sum_0^k a_j b_{k-j}. \quad (5.8)$$

Accordingly we obtain for the inverse Fourier transform $\sigma(x)$ the following expansion, presumably convergent for small x :

$$2\pi\sigma(x) = \sum_0^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(k + \frac{1}{2})} c_k x^{k-\frac{1}{2}}. \quad (5.9)$$

In the case $a = 0$ the first few coefficients are

$$\begin{array}{ll} b_0 = 0.337, & c_0 = 0.337, \\ b_1 = 0.254, & c_1 = 0.263, \\ b_2 = -0.614, & c_2 = -0.586, \\ b_3 = -1.803, & c_3 = -1.803, \end{array}$$

so that

$$2\pi\sigma(x) = 0.337 x^{-\frac{1}{2}} (1 + 1.56x - 2.32x^2 - 2.85x^3 \dots). \quad (5.10)$$

From this we obtain $\sigma(x)$ for a few x values (table II).

TABLE II

x	$\sigma(x)$	x	$\sigma(x)$
0.001	10.69	0.02	2.46
0.002	7.57	0.05	1.62
0.003	6.19	0.1	1.21
0.004	5.37	0.2	0.92
0.005	4.81	0.5	0.57
0.01	3.43		

§ 6. *Expressions for large $|a|$.* We shall consider the case $a > 0$ and $a < 0$ separately. If $a > 0$, we have from (5.2)

$$S^+(z) = \frac{e^{iaz}}{2\pi I_0(z)} - \frac{1}{2\pi H^+(z)} \sum_1^\infty \frac{h_k e^{-a\beta_k}}{(iz + \beta_k) J_1(\beta_k)}, \quad (6.1)$$

where

$$h_k = H^+(i\beta_k).$$

The inverse transformation of (6.1) gives

$$\sigma(x) = \sigma_0(x - a) + \frac{1}{2\pi} \sum_1^\infty \frac{h_k e^{-a\beta_k}}{J_1(\beta_k)} B(\beta_k, x), \quad (6.2)$$

where $\sigma_0(x - a)$ represents the surface charge of a double-infinite cylinder due to a point charge -1 at $x = a$, $r = 1$, and where the function $B(\beta, x)$ is the inverse Fourier transform of

$$[H^+(z)(-iz - \beta)]^{-1}$$

or

$$B(\beta, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{dt}{H^+(t)(-it - \beta)}. \quad (6.3)$$

If $a < 0$, we obtain from (5.1) by taking both sides of the negative imaginary axis as new path of integration

$$S^+(z) = \frac{1}{H^+(z)} \int_0^\infty e^{at} \frac{J_0(t)}{H^+(ti)} \frac{dt}{t - iz}, \quad (6.4)$$

from which

$$\sigma(x) = \int_0^\infty e^{at} \frac{J_0(t)}{H^+(ti)} B(-t, x) dt. \quad (6.5)$$

The expressions (6.2) and (6.5) are useful if $|a|$ is large. If, however, $|a|$ is small, we may proceed as follows. From (5.1) or (5.2) we obtain at once by inverse transformation

$$\sigma(x) = \sigma_0(x - a) + \frac{1}{2\pi^2 i} \int_L \frac{e^{-iux}}{H^+(u)} du \int_{-\infty}^{\infty} e^{iat} \frac{K_0(|t|)}{H^-(t)} \frac{dt}{t - u},$$

where L is a contour along both sides of the negative imaginary axis. This may be written as

$$\sigma(x) = \sigma_0(x - a) + \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iat} \frac{K_0(|t|)}{H^-(t)} B(-ti, x) dt, \quad (6.6)$$

where

$$B(\beta, x) = \frac{1}{2\pi} \int_L e^{-itx} \frac{dt}{H^+(t)(-it - \beta)}, \quad \arg \beta = -\frac{\pi}{2}. \quad (6.7)$$

The functions $B(\beta, x)$ which are used in (6.2), (6.5) and (6.6) may be reduced as follows

a. β real and positive:

$$\begin{aligned} B(\beta, x) &= \frac{1}{8\pi^2} \int_0^{\infty} e^{-itx} \frac{H^-(t)}{I_0(t) K_0(|t|)} \frac{dt}{-it - \beta} = \\ &= \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-itx} \left[\frac{I_0'(t)}{I_0(t)} - \frac{t}{|t|} \frac{K_0'(|t|)}{K_0(|t|)} \right] \frac{tH^-(t)}{-it - \beta} dt = \\ &= \frac{1}{4\pi} \sum_1^{\infty} \frac{\beta_j h_j}{\beta_j + \beta} e^{-\beta_j x} - \frac{1}{4\pi^2} \operatorname{Re} \int_0^{\infty} e^{-tx} \frac{K_0'(-ti)}{K_0(-ti)} \frac{tH^-(-ti)}{t + \beta} dt \end{aligned}$$

and finally

$$B(\beta, x) = \frac{1}{4\pi} \sum_1^{\infty} \frac{\beta_j h_j}{\beta_j + \beta} e^{-\beta_j x} + \frac{1}{2\pi^3} \int_0^{\infty} e^{-tx} \frac{H^+(ti)}{J_0^2(t) + Y_0^2(t)} \frac{dt}{t + \beta}. \quad (6.8)$$

b. β real and negative:

In a similar way we obtain

$$\begin{aligned} B(\beta, x) &= \frac{1}{4\pi} \sum_1^{\infty} \frac{\beta_j h_j}{\beta_j + \beta} e^{-\beta_j x} - \frac{\beta J_1(\beta)}{4\pi J_0(\beta)} H^+(-\beta i) e^{\beta x} + \\ &+ \frac{1}{2\pi^3} \int_0^{\infty} e^{-tx} \frac{H^+(ti)}{J_0^2(t) + Y_0^2(t)} \frac{dt}{t + \beta} \end{aligned} \quad (6.9)$$

c. β purely imaginary:

In this case the expression (6.8) may be used.

The expressions (6.8) and (6.9) are reasonable convergent if x is large. For small x we better take the expansion derived in the previous section.

§ 7. *Behaviour of $\sigma(x)$ for $x \rightarrow \infty$.* The behaviour of $\sigma(x)$ for $x \rightarrow \infty$ may be determined from the behaviour of $S^+(z)$ for $z \rightarrow 0$. From (5.2) we obtain for small z

$$S^+(z) = \frac{e^{iaz}}{2\pi I_0(z)} + \frac{1}{4\pi^2 i H^+(z)} \int_{L'} e^{iat} \frac{H^+(t)}{I_0(t)} \frac{dt}{t-z}, \quad (7.1)$$

where L' is the real axis with a semi-circular indentation at the origin which separates z from the poles $i\beta_k$, $k = 1, 2, \dots$ of $I_0^{-1}(t)$. Then for $z \rightarrow 0$ we obtain from (7.1) in view of (4.18)

$$2\pi S^+(z) = 1 - \frac{C}{2} \left(\pi \ln \frac{2}{z} \right)^{-\frac{1}{2}} + O \left[\left(\ln \frac{2}{z} \right)^{-3/2} \right], \quad (7.2)$$

where

$$C = - \frac{1}{2\pi i} \int_{L'} e^{iat} \frac{H^+(t)}{t I_0(t)} dt. \quad (7.3)$$

For $\sigma(x)$ we find accordingly

$$\sigma(x) = \frac{C}{8\pi^{\frac{1}{2}}} \frac{1}{x(\ln x)^{3/2}} + O[x^{-1} (\ln x)^{-5/2}]. \quad (7.4)$$

The coefficient C may be expressed as follows for $a > 0$:

$$C = \sum_{k=1}^{\infty} \frac{H^+(i\beta_k) e^{-a\beta_k}}{\beta_k J_1(\beta_k)}. \quad (7.5)$$

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- 4) Sparenberg, J. A., Proc. K. Ned. Akad. Wet. A **59** (1956) 29.
- 5) See ref.², § 15 and § 26.